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On Commutativity of a Semigroup which is a Semilattice of Commutative Semigroups

REIKICHI YOSHIDA

Department of Mathematics, Ritumeikan University, Kyoto, Japan

AND

MIYUKI YAMADA

Department of Mathematics, Shimane University, Matsue, Japan

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Let $P_1(G)$ and $P_2(G)$ be abstract properties pertaining to commutative semigroups G in the sense of Cohn [*3*]. $P_1(G)$ is said to be weaker than or equal to $P_2(G)$ and denoted by $P_1(G) \geqslant P_2(G)$ if and only if, for any commutative semigroup S, $P_1(G)$ is satisfied by S (i.e., $P_1(S)$ is true) whenever $P_2(G)$ is satisfied by S. If $P_1(G) \geqslant P_2(G)$ and $P_2(G) \geqslant P_1(G)$, then $P_1(G)$ and $P_2(G)$ are said to be equivalent and denoted by $P_1(G) \equiv P_2(G)$. If $P_1(G) \equiv P_2(G)$, we regard $P_1(G)$ and $P_2(G)$ as the same property. When S is a semigroup which is a semilattice of commutative semi-groups S_ξ, $\xi \in \chi$, S is not necessarily commutative. However, there is an abstract property $P(G)$ pertaining to commutative semigroups G, such that, any semigroup which is a semilattice of commutative semigroups with $P(G)$ is commutative. Such an abstract property $P(G)$ is called a fully c-invariant property (abbrev., f.c.i.-property). For example, it is well-known (e.g., see Clifford [*1*]) that the property $P(G)$, "G is a group", is an f.c.i.-property. There is no greatest (i.e., weakest) f.c.i.-property with respect to the ordering relation defined above, but there is a maximal f.c.i-property. Further, a maximal f.c.i.-property is not unique. The main purpose of this paper is to obtain maximal f.c.i.-properties, and some relevant results.

1. INTRODUCTION

A commutative idempotent semigroup Γ is called a *semilattice*. Define an ordering relation on Γ as follows:

(1.1) $\alpha \leqslant \beta$ if and only if $\alpha\beta = \beta\alpha = \beta$.

Then, it is obvious that Γ is a partially ordered set with respect to $\leqslant$. If $\alpha \leqslant \beta$ and $\alpha \neq \beta$, then we shall denote it by $a < \beta$. If Γ is a totally ordered set with respect to $\leqslant$, then Γ is called a *chain*. Now, let $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) be a collection of semigroups S_γ. Then, each S_γ is called the *γ-component* of this collection. If γ is not a minimal element of Γ (i.e., if there is an element $\alpha \in \Gamma$ such that $\alpha < \gamma$), then the corresponding S_γ is called a *multiple-component*. Let $S = \sum\{S_\gamma : \gamma \in \Gamma\}$ (hereafter, $\sum$ and $\dotplus$ denote disjoint sum). If $\circ$ is multiplication in S such that

(1.2) $S(\circ)$ is a semigroup, and each $S_\gamma(\gamma \in \Gamma)$ is embedded in $S(\circ)$, i.e., $x \circ y = xy$ for all $x, y \in S_\gamma$, and

(1.3) $S_\alpha \circ S_\beta \subset S_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$,

then the resulting system $S(\circ)$ is called a *composition of* $\{S_\gamma : \gamma \in \Gamma\}$ (*with respect to* Γ). Further, next we shall generalize this concept as follows: Let $\{S_\xi : \xi \in \chi\}$ (χ: a set) be a collection of semigroups S_ξ. Define multiplication $*$ in χ and multiplication $\circ$ in $S = \sum\{S_\xi : \xi \in \chi\}$ such that $\chi(*)$ is a semilattice [chain] and $S(\circ)$ is a composition of $\{S_\xi : \xi \in \chi(*)\}$. In this case, $S(\circ)$ is called a *semilattice* [*linear*] *composition of* $\{S_\xi : \xi \in \chi\}$. Let $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) be a collection of commutative semigroups S_γ. Then, sometimes there exists a composition $S(\circ)$ of $\{S_\gamma : \gamma \in \Gamma\}$ which is commutative. In this case, we shall call $S(\circ)$ a *commutative composition of* $\{S_\gamma : \gamma \in \Gamma\}$. Similarly if a semilattice [linear] composition $S(\circ)$ of a collection $\{S_\xi : \xi \in \chi\}$ (χ: a set) of commutative semigroups S_ξ is commutative, then $S(\circ)$ is called a *commutative semilattice* [*linear*] *composition of* $\{S_\xi : \xi \in \chi\}$. In general, for a given collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) of semigroups S_γ there is not necessarily a composition of $\{S_\gamma : \gamma \in \Gamma\}$ (see [*4*]). If there exists at least one composition of $\{S_\gamma : \gamma \in \Gamma\}$, then the collection $\{S_\gamma : \gamma \in \Gamma\}$ is said to be *composable*. If Γ is a chain, then it is well-known (e.g., see [*1*]) that $\{S_\gamma : \gamma \in \Gamma\}$ is necessarily composable. For any given collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) of commutative semigroups S_γ, a composition of $\{S_\gamma : \gamma \in \Gamma\}$ is (even if it exists) not necessarily commutative. This can be seen from the following simple example:

Let $\Gamma = \{\alpha, \beta\}$ ($\alpha\beta = \beta\alpha = \beta, \alpha \neq \beta$) be a chain, S_α a commutative semigroup, and S_β a null semigroup containing at least two elements. Let $S = S_\alpha \dotplus S_\beta$, and define multiplication $\circ$ in S as follows:

$$x \circ y = \begin{cases} xy & \text{if } x, y \in S_\alpha \text{ or } \in S_\beta, \\ y & \text{if } x \in S_\alpha, y \in S_\beta, \\ 0 & \text{if } x \in S_\beta, y \in S_\alpha, \end{cases}$$

where 0 is the zero element of S_β. Then $S(\circ)$ is a non-commutative composition of $\{S_\alpha, S_\beta\}$ with respect to Γ. In Section 2, we shall give a necessary and sufficient condition for a collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) of commutative semigroups S_γ to be composable. Further, in the case where $\{S_\gamma : \gamma \in \Gamma\}$ is composable, we shall give a method of construction of all compositions of $\{S_\gamma : \gamma \in \Gamma\}$. We also give a necessary and sufficient condition for $\{S_\gamma : \gamma \in \Gamma\}$ that every composition of $\{S_\gamma : \gamma \in \Gamma\}$ (if it exists) be necessarily commutative.

Let $P(G)$ be a proposition pertaining to commutative semigroups G. As in Cohn [3], $P(G)$ is said to be an *abstract property* (pertaining to commutative semigroups) if, and only if, $P(G)$ is invariant under isomorphism, i.e.,

(1.4) for any commutative semigroups S_1, S_2 such that $S_1 \simeq S_2$ (S_1 is isomorphic with S_2), $P(S_1)$ is true whenever $P(S_2)$ is true and vice-versa.

If $P(S)$ is true for a commutative semigroup S, then we shall say that S *satisfies* $P(G)$. In this case, we also say that S is a commutative semigroup *with* $P(G)$. For example, the properties "G is a group" and "G is cancellative" (pertaining to commutative semigroups G) are abstract properties. Let $P_1(G)$ and $P_2(G)$ be abstract properties. Then $P_1(G)$ and $P_2(G)$ are said to be *equivalent* (denoted by $P_1(G) = P_2(G)$) if the following (1.5) is fulfilled.

(1.5) For any commutative semigroup S, $P_1(S)$ is true if, and only if, $P_2(S)$ is true.

Hereafter, we shall consider $P_1(G)$ and $P_2(G)$ as the same property if they are equivalent. Define an ordering relation on the set $\mathfrak{P}$ of abstract properties as follows: Let $P_1(G)$ and $P_2(G)$ be abstract properties. $P_1(G) \leqslant P_2(G)$ if the following (1.6) is fulfilled:

(1.6) For every commutative semigroup S, $P_2(S)$ is true whenever $P_1(S)$ is true.

If $P_1(G) \leqslant P_2(G)$ and $P_1(G) \neq P_2(G)$, then the property $P_2(G)$ is said to be *weaker* than the property $P_1(G)$ and denoted by $P_1(G) < P_2(G)$. It is obvious that $\mathfrak{P}$ is a partially ordered set with respect to this relation $\leqslant$ (when we regard properties $P_1(G)$ and $P_2(G)$ as the same property if $P_1(G) = P_2(G)$).

Next, consider the following propositions concerning an abstract property $P(G)$:

(1.7) For any collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) of commutative semigroups S_γ, where each multiple-component S_α satisfies $P(G)$, every composition of $\{S_\gamma : \gamma \in \Gamma\}$ is commutative.

(1.8) For any collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) of commutative semigroups S_γ, where each multiple-component S_α satisfies $P(G)$, every composition of $\{S_\gamma : \gamma \in \Gamma\}$ (if it exists) is commutative.

If (1.7) or (1.8) is true for $P(G)$, then $P(G)$ is called a *linearly c-extensible property* (abbrev., l.c.e.-property) or *fully c-extensible property* (abbrev., f.c.e.-property) respectively. For example, the abstract property "G is a group" pertaining to commutative semigroup G is an f.c.e.-property. By the definitions an f.c.e.-property is clearly an l.c.e.-property, but the converse is not true (see Remark below). In Section 3, we shall prove the existence of the weakest l.c.e.-property and the weakest f.c.e.-property and try to determine these properties.

Next, consider also the following propositions concerning an abstract property $P(G)$:

(1.9) For any collection $\{S_\xi : \xi \in \chi\}$ (χ: a set) of commutative semigroups S_ξ, where each S_ξ satisfies $P(G)$, every linear composition of $\{S_\xi : \xi \in \chi\}$ is commutative.

(1.10) For any collection $\{S_\xi : \xi \in \chi\}$ (χ: a set) of commutative semigroups S_ξ, where each S_ξ satisfies $P(G)$, every semilattice composition of $\{S_\xi : \xi \in \chi\}$ is commutative.

If (1.9) or (1.10) is true for $P(G)$, then $P(G)$ is called a *linearly c-invariant property* (abbrev., l.c.i-property) or a *fully c-invariant property* (abbrev., f.c.i.-property) respectively. It is obvious from the definitions that an l.c.e. [f.c.e.]-property is an l.c.i. [f.c.i.]-property. In Section 4, we shall prove the existence of maximal l.c.i.-properties and maximal f.c.i.-properties and determine some of them.

Remark. Let $P_u(G)$ be an abstract property as follows:

(1.11) G is universal, i.e., $G^2 = G$.

Then it is easy to see that *universality* $P_u(G)$ is an l.c.e.-property (this will be shown later). Now, let T be a universal commutative semigroup which has a zero element 0 and whose annihilator A contains a non-zero element. The existence of such a semigroup T is easily seen from the following simple example: Let T^* be the semigroup consisting of all real numbers $\alpha > 1$ with respect to the usual multiplication. Then $I = \{\beta : \beta \geqslant 25, \beta \in T^*\}$ is an ideal of T^*. Let $T_1 = T^*/I$ be the Rees factor semigroup of T^* modulo $I : T_1 = (1, 25) \dotplus \{O\}$, where O is the zero element of T_1. Let $\Omega = \{(x, y) : x, y \in T_1, xy = O \text{ in } T_1\}$, and let $\Gamma = \Omega \backslash \{(x, y) : x, y \in (1, 25), x \times y = 25\}$ where $x \times y$ denotes the usual product of the real numbers x and y. Let $N = \{u, 0\}$ be a null semigroup of order 2, where 0 is the zero element of N. Let $T = N \dotplus T_1 \backslash O$, and define multiplication $\circ$ in T as follows: For $x, y \in T$,

$$x \circ y = \begin{cases} xy & \text{if } x, y \in T_1 \backslash O \text{ and } xy \neq O \text{ or if } x, y \in N, \\ 0 & \text{if } x \in N \text{ or } y \in N, \\ 0 & \text{if } x, y \in T_1 \backslash O, xy = O \text{ and } (x, y) \in \Gamma, \\ u & \text{if } x\ y \in T_1 \backslash O, xy = O \text{ and } (x, y) \notin \Gamma. \end{cases}$$

Then T is a universal commutative semigroup which has a zero element, and whose annihilator contains a non-zero element.

Now, let $L_3 = \{\alpha, \beta, \gamma\}$ be a semilattice such that $\alpha < \gamma$, $\beta < \gamma$, $\alpha \nleq \beta$ and $\beta \nleq \alpha$. Let S_α and S_β be infinite cyclic semigroups generated by a and b respectively: $S_\alpha = (a)$ and $S_\beta = (b)$. Let $S_\gamma = T$. Then $S = S_\alpha \dotplus S_\beta \dotplus S_\gamma$ becomes a non-commutative composition of $\{S_\alpha, S_\beta, S_\gamma\}$ with respect to L_3 by the multiplication $\circ$ defined as follows:

$$x \circ y = \begin{cases} xy & \text{if} \quad x, y \in S_\alpha, \quad \in S_\beta \quad \text{or} \quad \in S_\gamma, \\ u & \text{if} \quad x = a \quad \text{and} \quad y = b, \\ 0 & \text{otherwise}, \end{cases}$$

where 0 is the zero element of $T(=S_\gamma)$ and u is a fixed non-zero element contained in the annihilator of $T(=S_\gamma)$. Hence $P_u(G)$ is not an f.c.e.-property.

Throughout this paper, if $\{S_\xi : \xi \in \chi\}$ is a collection of commutative semigroups S_ξ, we shall denote elements of S_ξ by small letters a_ξ, b_ξ, c_ξ etc. having ξ as their subscripts.

2. Composition Theorems

Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) be a collection of commutative semigroups S_γ. For every pair (α, β) of $\alpha, \beta \in \Gamma$ with $\alpha \leqslant \beta$, let $\mathfrak{M}(\alpha, \beta)$ be the set of mappings of S_α into S_β. Let $C(\alpha, \beta) = \{S_\xi : \xi \in \Gamma,\ \alpha\xi = \beta\}$. Clearly $S_\beta \in C(\alpha, \beta)$. For every $S_\xi \in C(\alpha, \beta)$, let ψ_ξ, φ_ξ be (not necessarily distinct) two mappings of S_ξ into $\mathfrak{M}(\alpha, \beta)$. Put $\psi_\xi(a_\xi) = \bar{a}^{(\alpha,\beta)}$ and $\varphi_\xi(a_\xi) = \tilde{a}^{(\alpha,\beta)}$. Let $\mathfrak{M}_L(\Omega) = \mathfrak{M}_L(S_\gamma : \gamma \in \Gamma) = \{\bar{a}^{(\alpha,\beta)} : \alpha \leqslant \beta,\ \alpha, \beta \in \Gamma,\ a_\xi \in S_\xi,\ S_\xi \in C(\alpha, \beta)\}$, and $\mathfrak{M}_R(\Omega) = \mathfrak{M}_R(S_\gamma : \gamma \in \Gamma) = \{\tilde{a}_\xi^{(\alpha,\beta)} : \alpha \leqslant \beta,\ \alpha, \beta \in \Gamma,\ a_\xi \in S_\xi,\ S_\xi \in C(\alpha, \beta)\}$. If

$$(2.1) \quad \mathfrak{M}(\Omega) = \mathfrak{M}(S_\gamma : \gamma \in \Gamma) = \mathfrak{M}_L(\Omega) \dotplus \mathfrak{M}_R(\Omega)$$

satisfies the following condition (C), then $\mathfrak{M}(\Omega)$ is called a *set of composite factors on* Ω:

$$(\mathrm{C}) \begin{cases} (1) \ \bar{a}_\alpha^{(\beta,\alpha\beta)} \tilde{c}_\gamma^{(\alpha\beta,\alpha\beta\gamma)} = \tilde{c}_\gamma^{(\beta,\beta\gamma)} \bar{a}_\alpha^{(\beta\gamma,\alpha\beta\gamma)}, \\ (2) \ \bar{a}_\alpha^{(\alpha,\alpha)} = \tilde{a}_\alpha^{(\alpha,\alpha)} = \text{the inner translation } \rho_{a_\alpha} \text{ on } S_\alpha \\ \quad\ \text{induced by } a_\alpha, \\ (3) \ \bar{a}_\alpha^{(\beta,\alpha\beta)} \text{ and } \tilde{b}_\beta^{(\alpha,\alpha\beta)} \text{ are conjugate to each other} \\ \quad\ \text{in the following sense: } \bar{a}_\alpha^{(\beta,\alpha\beta)}(b_\beta) = \tilde{b}_\beta^{(\alpha,\alpha\beta)}(a_\alpha). \end{cases}$$

THEOREM 1. *Let* $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ *(Γ: a semilattice) be a collection of commutative semigroups* S_γ.

(i) *Ω is composable if and only if there exists a set of composite factors on Ω.*

(ii) *Let $\mathfrak{M}(\Omega)$ of* (2.1) *be a set of composite factors on Ω. Then $S = \Sigma\{S_\gamma : \gamma \in \Gamma\}$ becomes a composition $S(\circ)$ of Ω by the multiplication $\circ$ defined by*

(2.2) $\quad a_\alpha \circ b_\beta = \bar{a}_\alpha^{(\beta,\alpha\beta)}(b_\beta)\ (=\tilde{b}_\beta^{(\alpha,\alpha\beta)}(a_\alpha)).$

Further, every possible composition of Ω is found in this fashion.

Proof. (i) follows from (ii) and the definition of composability. Hence, we prove only (ii). Now let $\mathfrak{M}(\Omega)$ of (2.1) be a set of composite factors on Ω. For a_α, b_β, $c_\gamma \in S$, $(a_\alpha \circ b_\beta) \circ c_\gamma = \bar{a}_\alpha^{(\beta,\alpha\beta)}(b_\beta) \circ c_\gamma = \bar{a}_\alpha^{(\beta,\alpha\beta)}\, \tilde{c}_\gamma^{(\alpha\beta,\alpha\beta\gamma)}(b_\beta) = \tilde{c}_\gamma^{(\beta,\beta\gamma)}\bar{a}_\alpha^{(\beta\gamma,\alpha\beta\gamma)}(b_\beta) = \bar{a}_\alpha^{(\beta\gamma,\alpha\beta\gamma)}(\tilde{c}_\gamma^{(\beta,\beta\gamma)}(b_\beta)) = \bar{a}_\alpha^{(\beta\gamma,\alpha\beta\gamma)}(b_\beta \circ c_\gamma) = a_\alpha \circ (b_\beta \circ c_\gamma)$. Since $a_\alpha \circ b_\beta \in S_{\alpha\beta}$ for all $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$ and since $a_\alpha \circ b_\alpha = a_\alpha b_\alpha$ for all a_α, $b_\alpha \in S_\alpha$, $S(\circ)$ is a composition of Ω. Conversely, let $S(\bigcirc) = \Sigma\{S_\gamma : \gamma \in \Gamma\}$ be a composition of Ω. For $\alpha, \beta, \xi \in \Gamma$ with $\alpha \leqslant \beta$, $\alpha\xi = \beta$ and for $a_\xi \in S_\xi$, define mappings $\bar{a}_\xi^{(\alpha,\beta)} : S_\alpha \to S_\beta$ and $\tilde{a}_\xi^{(\alpha,\beta)} : S_\alpha \to S_\beta$ as follows: $\bar{a}_\xi^{(\alpha,\beta)}(b_\alpha) = a_\xi \bigcirc b_\alpha$ and $\tilde{a}_\xi^{(\alpha,\beta)}(b_\alpha) = b_\alpha \bigcirc a_\xi$. Then it is easy to see that $\mathfrak{M}(\Omega) = \{\bar{a}_\xi^{(\alpha,\beta)} : \alpha \leqslant \beta,\ a_\xi \in S_\xi \text{ with } \alpha\xi = \beta\} \dotplus \{\tilde{a}_\xi^{(\alpha,\beta)} : \alpha \leqslant \beta,\ a_\xi \in S_\xi \text{ with } \alpha\xi = \beta\}$ satisfies (1)–(3) of the condition (C) in (2.1) and hence is a set of composite factors on Ω. Let $S(\circ)$ be the composition of Ω determined by this $\mathfrak{M}(\Omega)$ and (2.2). Then, $a_\alpha \circ b_\beta = \bar{a}_\alpha^{(\beta,\alpha\beta)}(b_\beta) = a_\alpha \bigcirc b_\beta$. Hence $S(\circ) = S(\bigcirc)$.

The composition $S(\circ)$ in (ii) of Theorem 1 is called the composition of Ω *induced* by $\mathfrak{M}(\Omega)$.

COROLLARY 1. *Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) be a collection of commutative semigroups S_γ, and $\mathfrak{M}(\Omega)$ of* (2.1) *a set of composite factors on Ω. Then, the composition $S(\circ)$ of Ω induced by $\mathfrak{M}(\Omega)$ is non-commutative if and only if the following condition is satisfied*:

(2.3) $\quad \bar{a}_\alpha^{(\beta,\alpha\beta)} \neq \tilde{a}_\alpha^{(\beta,\alpha\beta)}$ *for some* $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$.

Proof. If $\bar{a}_\alpha^{(\beta,\alpha\beta)} \neq \tilde{a}_\alpha^{(\beta,\alpha\beta)}$, then there exists $b_\beta \in S_\beta$ such that $\bar{a}_\alpha^{(\beta,\alpha\beta)}(b_\beta) \neq \tilde{a}_\alpha^{(\beta,\alpha\beta)}(b_\beta)$. Hence $a_\alpha \circ b_\beta \neq b_\beta \circ a_\alpha$ in $S(\circ)$. Conversely, suppose that the composition $S(\circ)$ of Ω induced by $\mathfrak{M}(\Omega)$ is non-commutative. Then $a_\xi \circ b_\eta \neq b_\eta \circ a_\xi$ for some $a_\xi \in S_\xi$, $b_\eta \in S_\eta$, $\xi, \eta \in \Gamma$. Hence $\bar{a}_\xi^{(\eta,\xi\eta)}(b_\eta) = a_\xi \circ b_\eta \neq b_\eta \circ a_\xi = \tilde{a}_\xi^{(\eta,\xi\eta)}(b_\eta)$. This implies that $\bar{a}_\xi^{(\eta,\xi\eta)} \neq \tilde{a}_\xi^{(\eta,\xi\eta)}$.

COROLLARY 2. *Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) be a collection of commutative semigroups S_γ. Then, every composition of Ω is commutative if and only if there is no set, $\mathfrak{M}(\Omega)$ of* (2.1), *of composite factors on Ω which satisfies the condition* (2.3).

Now, as a special case, we consider a collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ of commutative semigroups S_γ having a *chain* Γ as its index set. Let $\mathfrak{M}(\Omega)$ of (2.1) be a set of composite factors on Ω.

Then, we have the following lemmas:

LEMMA 1. *For* $\alpha, \beta \in \Gamma$ *with* $\alpha \leqslant \beta$, *each of* $\bar{a}_\alpha^{(\beta,\alpha\beta)}(=\bar{a}_\alpha^{(\beta,\beta)})$ *and* $\tilde{a}_\alpha^{(\beta,\alpha\beta)}(=\tilde{a}_\alpha^{(\beta,\beta)}$ *is a translation on* S_β.

Proof. It follows from (1) of (C) in (2.1) that $\bar{a}_\alpha^{(\beta,\beta)}(b_\beta d_\beta) = \bar{a}_\alpha^{(\beta,\beta)}(\bar{d}_\beta^{(\beta,\beta)}(b_\beta)) = \bar{d}_\beta^{(\beta,\beta)}\bar{a}_\alpha^{(\beta,\beta)}(b_\beta) = \bar{a}_\alpha^{(\beta,\beta)}\bar{d}_\beta^{(\beta,\beta)}(b_\beta) = \bar{d}_\beta^{(\beta,\beta)}(\bar{a}_\alpha^{(\beta,\beta)}(b_\beta)) = \bar{a}_\alpha^{(\beta,\beta)}(b_\beta)\, d_\beta$. Hence, $\bar{a}_\alpha^{(\alpha,\alpha\beta)}(=\bar{a}_\alpha^{(\beta,\beta)})$ is a translation on S_β. Similarly, it can be easily proved that $\tilde{a}_\alpha^{(\beta,\alpha\beta)}(=\tilde{a}_\alpha^{(\beta,\beta)})$ is also a translation on S_β.

LEMMA 2. *For* $\alpha, \beta \in \Gamma$ *with* $\alpha \geqslant \beta$, $\bar{a}_\alpha^{(\beta,\alpha)}(b_\beta) = \tilde{b}_\beta^{(\alpha,\alpha)}(a_\alpha)$ *and* $\tilde{b}_\alpha^{(\beta,\alpha)}(a_\beta) = \bar{a}_\beta^{(\alpha,\alpha)}(b_\alpha)$.

Proof. This follows from (3) of (C) in (2.1).

Now, put $\bar{a}_\alpha^{(\beta,\beta)} = \rho_{a_\alpha,\beta}$ and $\tilde{a}_\alpha^{(\beta,\beta)} = \sigma_{a_\alpha,\beta}$ for $\alpha \leqslant \beta$. Then

LEMMA 3. $\rho_{a_\alpha,\alpha} = \sigma_{a_\alpha,\alpha} =$ *the inner translation* ρ_{a_α} *on* S_α *induced by* a_α.

Proof. Obvious.

LEMMA 4. $\rho_{a_\alpha,\gamma}\sigma_{b_\beta,\gamma} = \sigma_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}$ *if* $\alpha \leqslant \gamma$, $\beta \leqslant \gamma$.

Proof. $\rho_{a_\alpha,\gamma}\sigma_{b_\beta,\gamma} = \bar{a}_\alpha^{(\gamma,\gamma)}\tilde{b}_\beta^{(\gamma,\gamma)} = \tilde{b}_\beta^{(\gamma,\gamma)}\bar{a}_\alpha^{(\gamma,\gamma)}$ (by (1) of (C) in (2.1)) $=\sigma_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}$.

LEMMA 5.

$$\rho_{b_\beta,}\, \rho_{a_\alpha,\gamma} = \begin{cases} \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma} & \text{if } \alpha \leqslant \beta \leqslant \gamma, \\ \rho_{\sigma_{b_\beta,\alpha}(a_\alpha),\gamma} & \text{if } \beta \leqslant \alpha \leqslant \gamma. \end{cases}$$

Proof. $\rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma} = \bar{b}_\beta^{(\gamma,\gamma)}\bar{a}_\alpha^{(\gamma,\gamma)}$. (The case $\alpha \leqslant \beta \leqslant \gamma$) $\rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}(c_\gamma) = \bar{b}_\beta^{(\gamma,\gamma)}\bar{a}_\alpha^{(\gamma,\gamma)}(c_\gamma) = \bar{a}_\alpha^{(\gamma,\gamma)}(\bar{b}_\beta^{(\gamma,\gamma)}(c_\gamma)) = \bar{a}_\alpha^{(\gamma,\gamma)}(\tilde{c}_\gamma^{(\beta,\gamma)}(b_\beta))$ (by Lemma 2) $= \tilde{c}_\gamma^{(\beta,\gamma)}\bar{a}_\alpha^{(\gamma,\gamma)}(b_\beta) = \bar{a}_\alpha^{(\beta,\beta)}\tilde{c}_\gamma^{(\beta,\gamma)}(b_\beta)$ (by (1) of (C) in (2.1)) $= \tilde{c}_\gamma^{(\beta,\gamma)}(\bar{a}_\alpha^{(\ ,\beta)}(b_\beta)) = \overline{\bar{a}_\alpha^{(\beta,\beta)}(b_\beta)}^{(\gamma,\gamma)}(c_\gamma) = \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma}(c_\gamma)$. Hence, $\rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma} = \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma}$.

(The case $\beta \leqslant \alpha \leqslant \gamma$) $\rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}(c_\gamma) = \bar{b}_\beta^{(\gamma,\gamma)}\bar{a}_\alpha^{(\gamma,\gamma)}(c_\gamma) = \bar{a}_\alpha^{(\gamma,\gamma)}(\bar{b}_\beta^{(\gamma,\gamma)}(c_\gamma)) = \bar{a}_\alpha^{(\gamma,\gamma)}(\tilde{c}_\gamma^{(\beta,\gamma)}(b_\beta))$ (by Lemma 2) $= \tilde{c}_\gamma^{(\beta,\gamma)}\bar{a}_\alpha^{(\gamma,\gamma)}(b_\beta) = \bar{a}_\alpha^{(\beta,\alpha)}\tilde{c}_\gamma^{(\alpha,\gamma)}(b_\beta)$ (by (1) of (C) in (2.1)) $= \tilde{c}_\gamma^{(\alpha,\gamma}(\bar{a}_\alpha^{(\beta,\alpha)}(b_\beta)) = \tilde{c}_\gamma^{(\alpha,\gamma)}(\tilde{b}_\beta^{(\alpha,\alpha)}(a_\alpha))$ (by Lemma 2) $= \overline{\tilde{b}_\beta^{(\alpha,\alpha)}(a_\alpha)}^{(\gamma,\gamma)}(c_\gamma)$ (by Lemma 2) $= \rho_{\tilde{b}_\beta^{(\alpha,\alpha)}(a_\alpha),\gamma}(c_\gamma) = \rho_{\sigma_{b_\beta,\alpha}(a_\alpha),\gamma}(c_\gamma)$.

Hence, $\rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma} = \rho_{\sigma_{b_\beta,\alpha}(a_\alpha),\gamma}$.

LEMMA 6.

$$\sigma_{b_\beta,\gamma}\sigma_{a_\alpha,\gamma} = \begin{cases}\sigma_{\sigma_{a_\alpha,\beta}(b_\beta),\gamma} & \text{if} \quad \alpha \leqslant \beta \leqslant \gamma, \\ \sigma_{\rho_{b_\beta,\alpha}(a_\alpha),\gamma} & \text{if} \quad \beta \leqslant \alpha \leqslant \gamma.\end{cases}$$

Proof. This can be proved by an analogous method to the proof of Lemma 5.

By Lemmas 3-6, we obtain the following result: Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ, $\mathfrak{M}(\Omega)$ of (2.1) a set of composite factors on Ω. Let $S(\circ)$ be the composition of $\{S_\gamma : \gamma \in \Gamma\}$ induced by $\mathfrak{M}(\Omega)$.

Then, there exists a system

(2.4) $\mathfrak{S}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqslant \beta,\ \alpha, \beta \in \Gamma\} \dotplus \{\sigma_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqslant \beta,$ $\alpha, \beta \in \Gamma\}$, where $\rho_{a_\alpha,\beta}$ and $\sigma_{a_\alpha,\beta}$ are mappings of S_β into S_β, such that

$$(\mathrm{T})\begin{cases}(1)\ \rho_{a_\alpha,\beta} \text{ and } \sigma_{a_\alpha,\beta} \text{ are translations on } S_\beta, \\ (2)\ \rho_{a_\alpha,\alpha} = \sigma_{a_\alpha,\alpha} = \text{the inner translation } \rho_{a_\alpha} \text{ on } S_\alpha \\ \qquad \text{induced by } a_\alpha, \\ (3)\ \rho_{a_\alpha,\gamma}\sigma_{b_\beta,\gamma} = \sigma_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}, \\ (4)\ \rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma} = \begin{cases}\rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma} & \text{if} \quad \alpha \leqslant \beta \leqslant \gamma, \\ \rho_{\sigma_{b_\beta,\alpha}(a_\alpha),\gamma} & \text{if} \quad \beta \leqslant \alpha \leqslant \gamma,\end{cases} \\ (5)\ \sigma_{b_\beta,\gamma}\sigma_{a_\alpha,\gamma} = \begin{cases}\sigma_{\sigma_{a_\alpha,\beta}(b_\beta),\gamma} & \text{if} \quad \alpha \leqslant \beta \leqslant \gamma, \\ \sigma_{\rho_{b_\beta,\alpha}(a_\alpha),\gamma} & \text{if} \quad \beta \leqslant \alpha \leqslant \gamma.\end{cases}\end{cases}$$

Further, the multiplication $\circ$ in $S(\circ)$ is represented by

$$(\mathrm{P}) \quad a_\alpha \circ b_\beta = \begin{cases}\bar{a}_\alpha^{(\beta,\beta)}(b_\beta) = \rho_{a_\alpha,\beta}(b_\beta) & \text{if} \quad \alpha \leqslant \beta, \\ \tilde{b}_\beta^{(\alpha,\alpha)}(a_\alpha) = \sigma_{b_\beta,\alpha}(a_\alpha) & \text{if} \quad \alpha \geqslant \beta.\end{cases}$$

In general, let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ. For each pair (a_α, β), where $a_\alpha \in S_\alpha$, $\alpha \leqslant \beta$ and $\alpha, \beta \in \Gamma$, let $\rho_{a_\alpha,\beta}$ and $\sigma_{a_\alpha,\beta}$ be (not necessarily distinct) two mappings of S_β into S_β. If $\mathfrak{S}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqslant \beta,\ \alpha, \beta \in \Gamma\} \dotplus \{\sigma_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqslant \beta,\ \alpha, \beta \in \Gamma\}$ satisfies (T) of (2.4), then $\mathfrak{S}(\Omega)$ is called a *factor set of translations on Ω*. From this definition and the above-mentioned result, we can conclude as follows: Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ. If $S(\circ) = \Sigma\{S_\gamma : \gamma \in \Gamma\}$ is a composition of $\Omega = \{S_\gamma : \gamma \in \Gamma\}$, then there exists a factor set of translations on Ω, say $\mathfrak{S}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqslant \beta,$ $\alpha, \beta \in \Gamma\} \dotplus \{\sigma_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqslant \beta,\ \alpha, \beta \in \Gamma\}$, and $\circ$ in $S(\circ)$ is represented by (P).

Conversely, let $\mathfrak{S}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqslant \beta, \alpha, \beta \in \Gamma\} \dotplus \{\sigma_{a_\alpha,\beta} : a_\alpha \in S_\alpha,$ $\alpha \leqslant \beta,\ \alpha, \beta \in \Gamma\}$ (Γ: a chain) be a factor set of translations on a collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ of commutative semigroups S_γ. Next, we prove that

$S = \Sigma\{S_\gamma : \gamma \in \Gamma\}$ becomes a composition of Ω by the multiplication $\circ$ given by (P). Let $S(\circ)$ be the set S in which multiplication $\circ$ is defined by (P). $a_\alpha \circ b_\alpha = \rho_{a_\alpha,\alpha}(b_\alpha) = a_\alpha b_\alpha$. Since $S_\alpha \circ S_\beta \subset S_\beta$ if $\alpha \leqslant \beta$ and since $S_\alpha \circ S_\beta \subset S_\alpha$ if $\alpha \geqslant \beta$, in the both cases $S_\alpha \circ S_\beta \subset S_{\alpha\beta}$. Hence to prove $S(\circ)$ to be a composition of Ω, it is sufficient to prove the associativity of $S(\circ)$. Let α, β, γ be elements of Γ such that $\alpha \leqslant \beta \leqslant \gamma$.

(i) $(a_\alpha \circ b_\beta) \circ c_\gamma = \rho_{a_\alpha,\beta}(b_\beta) \circ c_\gamma = \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma}(c_\gamma)$. On the other hand, $a_\alpha \circ (b_\beta \circ c_\gamma) = a_\alpha \circ (\rho_{b_\beta,\gamma}(c_\gamma)) = \rho_{a_\alpha,\gamma}(\rho_{b_\beta,\gamma}(c_\gamma)) = \rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}(c_\gamma)$. Thus by (4) of (T) in (2.4), $(a_\alpha \circ b_\beta) \circ c_\gamma = \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma}(c_\gamma) = \rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}(c_\gamma) = a_\alpha \circ (b_\beta \circ c_\gamma)$.

(ii) $(b_\beta \circ a_\alpha) \circ c_\gamma = \sigma_{a_\alpha,\beta}(b_\beta) \circ c_\gamma = \rho_{\sigma_{a_\alpha,\beta}(b_\beta),\gamma}(c_\gamma)$, while $b_\beta \circ (a_\alpha \circ c_\gamma) = b_\beta \circ (\rho_{a_\alpha,\gamma}(c_\gamma)) = \rho_{b_\beta,\gamma}(\rho_{a_\alpha,\gamma}(c_\gamma)) = \rho_{a_\alpha,\gamma}\rho_{b_\beta,\gamma}(c_\gamma)$. Hence by (4) of (T) in (2.4), $(b_\beta \circ a_\alpha) \circ c_\gamma = \rho_{\sigma_{a_\alpha,\beta}(b_\beta),\gamma}(c_\gamma) = \rho_{a_\alpha,\gamma}\rho_{b_\beta,\gamma}(c_\gamma) = b_\beta \circ (a_\alpha \circ c_\gamma)$.

(iii) $(a_\alpha \circ c_\gamma) \circ b_\beta = \rho_{a_\alpha,\gamma}(c_\gamma) \circ b_\beta = \sigma_{b_\beta,\gamma}(\rho_{a_\alpha,\gamma}(c_\gamma)) = \rho_{a_\alpha,\gamma}\sigma_{b_\beta,\gamma}(c_\gamma)$, while $a_\alpha \circ (c_\gamma \circ b_\beta) = a_\alpha \circ (\sigma_{b_\beta,\gamma}(c_\gamma)) = \rho_{a_\alpha,\gamma}(\sigma_{b_\beta,\gamma}(c_\gamma)) = \sigma_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}(c_\gamma)$. Hence by (3) of (T) in (2.4), $(a_\alpha \circ c_\gamma) \circ b_\beta = a_\alpha \circ (c_\gamma \circ b_\beta)$.

(iv) $(c_\gamma \circ a_\alpha) \circ b_\beta = \sigma_{a_\alpha,\gamma}(c_\gamma) \circ b_\beta = \sigma_{b_\beta,\gamma}(\sigma_{a_\alpha,\gamma}(c_\gamma)) = \sigma_{a_\alpha,\gamma}\sigma_{b_\beta,\gamma}(c_\gamma)$, while $c_\gamma \circ (a_\alpha \circ b_\beta) = c_\gamma \circ (\rho_{a_\alpha,\beta}(b_\beta)) = \sigma_{\rho_{a_\alpha,\beta}(b_\beta),\gamma}(c_\gamma)$. Hence by (5) of (T) in (2.4), $(c_\gamma \circ a_\alpha) \circ b_\beta = c_\gamma \circ (a_\alpha \circ b_\beta)$.

(v) $(b_\beta \circ c_\gamma) \circ a_\alpha = \rho_{b_\beta,\gamma}(c_\gamma) \circ a_\alpha = \sigma_{a_\alpha,\gamma}(\rho_{b_\beta,\gamma}(c_\gamma)) = \rho_{b_\beta,\gamma}\sigma_{a_\alpha,\gamma}(c_\gamma)$, while $b_\beta \circ (c_\gamma \circ a_\alpha) = b_\beta \circ (\sigma_{a_\alpha,\gamma}(c_\gamma)) = \rho_{b_\beta,\gamma}(\sigma_{a_\alpha,\gamma}(c_\gamma)) = \sigma_{a_\alpha,\gamma}\rho_{b_\beta,\gamma}(c_\gamma)$. Hence by (3) of (T) in (2.4), $(b_\beta \circ c_\gamma) \circ a_\alpha = b_\beta \circ (c_\gamma \circ a_\alpha)$.

(vi) $(c_\gamma \circ b_\beta) \circ a_\alpha = \sigma_{b_\beta,\gamma}(c_\gamma) \circ a_\alpha = \sigma_{a_\alpha,\gamma}(\sigma_{b_\beta,\gamma}(c_\gamma)) = \sigma_{b_\beta,\gamma}\sigma_{a_\alpha,\gamma}(c_\gamma)$, while $c_\gamma \circ (b_\beta \circ a_\alpha) = c_\gamma \circ (\sigma_{a_\alpha,\beta}(b_\beta)) = \sigma_{\sigma_{a_\alpha,\beta}(b_\beta),\gamma}(c_\gamma)$. Hence by (5) of (T) in (2.4), $(c_\gamma \circ b_\beta) \circ a_\alpha = c_\gamma \circ (b_\beta \circ a_\alpha)$.

In all cases, the associativity of the multiplication $\circ$ in $S(\circ)$ can be proved.

Summerizing the results above, we obtain the following

THEOREM 2. *Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ. Let $\mathfrak{S}(\Omega)$ of* (2.4) *be a factor set of translations on Ω. Then $S = \Sigma\{S_\gamma : \gamma \in \Gamma\}$ becomes a composition $S(\circ)$ of Ω by the multiplication $\circ$ defined by* (P). *Further, every composition of Ω is found in this fashion.*

$S(\circ)$ in Theorem 2 is called the composition of Ω *induced* by $\mathfrak{S}(\Omega)$. From Theorem 2, we obtain immediately the following

COROLLARY 1. *Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ, and $\mathfrak{S}(\Omega)$ of* (2.4) *a factor set of translations on Ω. Then, the composition $S(\circ)$ of Ω induced by $\mathfrak{S}(\Omega)$ is non-commutative if, and only if,*

(2.5) $\rho_{a_\alpha,\beta} \neq \sigma_{a_\alpha,\beta}$ *for some* $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$, $\alpha < \beta$.

Proof. Obvious.

COROLLARY 2. *Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ. Every composition of Ω is commutative if and only if there is no factor set, $\mathfrak{S}(\Omega)$ of* (2.4), *of translations on Ω which satisfies* (2.5).

Proof. Obvious from Corollary 1.
In the case where every composition of a collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) of commutative semigroups S_γ is commutative, we have another construction theorem for the compositions of Ω which is somewhat simpler than Theorem 2:

THEOREM 3. *Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ every composition of which is commutative. Let $S = \Sigma\{S_\gamma : \gamma \in \Gamma\}$. For each pair (a_α, β), where $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ and $\alpha \leqslant \beta$, let $\rho_{a_\alpha,\beta}$ be a mapping of S_β into S_β. Let $\bar{\mathfrak{S}}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha, \alpha \leqslant \beta, \alpha, \beta \in \Gamma\}$. If $\bar{\mathfrak{S}}(\Omega)$ satisfies the condition*

$$(\bar{\mathrm{T}}) \begin{cases} (1) & \rho_{a_\alpha,\beta} \text{ is a translation on } S_\beta, \\ (2) & \rho_{a_\alpha,\alpha} = \text{the inner translation } \rho_{a_\alpha} \text{ on } S_\alpha \text{ induced by } a_\alpha, \\ (3) & \rho_{a_\alpha,\gamma}\rho_{b_\beta,\gamma} = \rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma} = \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma} \text{ if } \alpha \leqslant \beta \leqslant \gamma, \end{cases}$$

then S becomes a composition $S(\circ)$ of Ω by the multiplication $\circ$ defined by

$$(\bar{\mathrm{P}}) \quad a_\alpha \circ b_\beta = b_\beta \circ a_\alpha = \rho_{a_\alpha,\beta}(b_\beta) \qquad \text{if} \quad \alpha \leqslant \beta.$$

Further, every composition of Ω is found in this fashion.

Proof. The first half of the theorem: Let $\sigma_{a_\alpha,\beta} = \rho_{a_\alpha,\beta}$, and put $\mathfrak{S}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha, \alpha \leqslant \beta, \alpha, \beta \in \Gamma\} \dot{+} \{\sigma_{a_\alpha,\beta} : a_\alpha \in S_\alpha, \alpha \leqslant \beta, \alpha, \beta \in \Gamma\}$. Then $\mathfrak{S}(\Omega)$ is clearly a factor set of translations on Ω. Hence, S becomes a composition of Ω by the multiplication $\circ$ defined by (P). On the other hand, in this case it is easy to see that (P) means $(\bar{\mathrm{P}})$.

The second half of the theorem: Suppose that $S(\circ)$ is a composition of $\{S_\gamma : \gamma \in \Gamma\}$. Then, by the assumption, $S(\circ)$ is commutative. For any $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ with $\alpha \leqslant \beta$, define a mapping $\rho_{a_\alpha,\beta} : S_\beta \to S_\beta$ by $\rho_{a_\alpha,\beta}(b_\beta) = a_\alpha \circ b_\beta = b_\beta \circ a_\alpha$. Then it is easy to see that $\bar{\mathfrak{S}}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha, \alpha \leqslant \beta, \alpha, \beta \in \Gamma\}$ satisfies $(\bar{\mathrm{T}})$ and the multiplication $\circ$ in $S(\circ)$ is given by $(\bar{\mathrm{P}})$.

Next, we present some results concerning a factor set of translations on a collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) of commutative semigroups S_γ.

LEMMA 7. *Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ. Let $\mathfrak{S}(\Omega)$ of* (2.4) *be a factor set of translations on Ω which satisfies* (T) *in* (2.4). *Then,*

(i) *for any $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ with $\alpha \leqslant \beta$, $y_\beta\rho_{a_\alpha,\beta}(x_\beta) = \sigma_{a_\alpha,\beta}(y_\beta)\, x_\beta$, i.e., $\rho_{a_\alpha,\beta}$ and $\sigma_{a_\alpha,\beta}$ are linked, and*

(ii) $\rho_{a_\alpha,\beta} \mid S_\beta^2 = \sigma_{a_\alpha,\beta} \mid S_\beta^2$.[1]

[1] $\rho_{a_\alpha,\beta} \mid S_\beta^2$ denotes the restriction of $\rho_{a_\alpha,\beta}$ to S_β^2.

Proof. $y_\beta \rho_{a_\alpha,\beta}(x_\beta) = \rho_{y_\beta,\beta}(\rho_{a_\alpha,\beta}(x_\beta))$ (by (2) of (T)) $= \rho_{a_\alpha,\beta}\rho_{y_\beta,\beta}(x_\beta) = \rho_{\sigma_{a_\alpha,\beta}(y_\beta),\beta}(x_\beta)$ (by (4) of (T)) $= \sigma_{a_\alpha,\beta}(y_\beta)\, x_\beta$ (by (2) of (T)).

(ii) $\rho_{a_\alpha,\beta}(x_\beta y_\beta) = \rho_{a_\alpha,\beta}(x_\beta)\, y_\beta = y_\beta \rho_{a_\alpha,\beta}(x_\beta) = \sigma_{a_\alpha,\beta}(y_\beta)\, x_\beta$ (by (i)) $= x_\beta \sigma_{a_\alpha,\beta}(y_\beta) = \sigma_{a_\alpha,\beta}(x_\beta y_\beta)$. Hence $\rho_{a_\alpha,\beta} \mid S_\beta^2 = \sigma_{a_\alpha,\beta} \mid S_\beta^2$.

A commutative semigroup S is said to be *reductive* if it satisfies the following abstract property $P_r(G)$:

Reductivity $P_r(G)$: $ax = bx$ for all $x \in G$ implies $a = b$.

LEMMA 8. *Let Ω and $\mathfrak{S}(\Omega)$ be as in Lemma 7. If each of the multiple-components of Ω is universal or reductive, then $\rho_{a_\alpha,\beta} = \sigma_{a_\alpha,\beta}$ for any $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ with $\alpha \leqslant \beta$.*

Proof. If $\alpha = \beta$, then clearly $\rho_{a_\alpha,\beta} = \sigma_{a_\alpha,\beta} = \rho_{a_\alpha}$. Suppose that $\alpha < \beta$. If S_β is universal, then $\rho_{a_\alpha,\beta} = \rho_{a_\alpha,\beta} \mid S_\beta^2 = \sigma_{a_\alpha,\beta} \mid S_\beta^2 = \sigma_{a_\alpha,\beta}$. If S_β is reductive, then by Lemma 7 $\rho_{a_\alpha,\beta}(a_\beta b_\beta) = \sigma_{a_\alpha,\beta}(a_\beta b_\beta)$ for any a_β, b_β. Hence $\rho_{a_\alpha,\beta}(a_\beta)\, b_\beta = \sigma_{a_\alpha,\beta}(a_\beta)\, b_\beta$ for all $b_\beta \in S_\beta$. By the reductivity of S_β, $\rho_{a_\alpha,\beta}(a_\beta) = \sigma_{a_\alpha,\beta}(a_\beta)$ for all $a_\beta \in S_\beta$. This implies that $\rho_{a_\alpha,\beta} = \sigma_{a_\alpha,\beta}$.

By using Lemma 8 and Corollary 2 to Theorem 2, we obtain

COROLLARY. *Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) be a collection of commutative semigroups S_γ. If each of the multiple-components of Ω is universal or reductive, then every composition of Ω is commutative.*

Proof. Obvious.

Remark. This result will be more generalized in the next section.

3. THE WEAKEST F.C.E. [L.C.E.]-PROPERTY

In this section, we investigate l.c.e.-properties and f.c.e.-properties.

Let us consider the following abstract property $P_q(G)$ pertaining to commutative semigroups G:

(3.1) There is no system $\{\sigma, \rho\}$ of distinct two translations on G such that (1) $\sigma\rho = \rho\sigma$ and (2) $\sigma \mid G^2 = \rho \mid G^2$.

This property $P_q(G)$ is called *quasi-reductivity*. As is shown later, reductivity implies quasi-reductivity. However, the converse is not true.

LEMMA 9. *Let $\{\sigma, \rho\}$ be a system of distinct two translations σ, ρ on a commutative semigroup S such that $\sigma\rho = \rho\sigma$ and $\sigma \mid S^2 = \rho \mid S^2$. Then there exist distinct two elements $x, y \in S$ and a prime element[2] $t \in S$ such that* (1) *$xa = ya$ for all $a \in S$ and* (2) *$\sigma(t) = x$ and $\rho(t) = y$.*

[2] An element of $S \backslash S^2$ is called prime.

Proof. Since $\sigma \mid S^2 = \rho \mid S^2$ and $\sigma \neq \rho$, $\sigma(t) \neq \rho(t)$ for some $t \in S \backslash S^2$. It is obvious that t is a prime element of S. Putting $\sigma(t) = x$ and $\rho(t) = y$, we have $xa = \sigma(t)\, a = \sigma(ta) = \rho(t)\, a = ya$ for all $a \in S$.

Example. Let $S = \{a, a^2, \ldots, a^n\}$ be a cyclic semigroup of order n such that $n \geqslant 2$, $a^{n-1} \neq a^n$ and $aa^n = a^n$. Define mappings ρ, $\sigma : S \to S$ as follows: $\rho(a) = a^{n-1}$, $\rho(a^i) = a^n$ if $i > 1$; and $\sigma(a^i) = a^n$ for all i. Then ρ, σ are translations on S such that $\sigma\rho = \rho\sigma$ and $\sigma \mid S^2 = \rho \mid S^2$. Hence, of course, S is not quasi-reductive.

By using Lemma 9, we can prove the following theorem:

THEOREM 4. *$P_q(G)$ is the weakest* l.c.e.*-property.*

Proof. At first, we prove that $P_q(G)$ is an l.c.e.-property. Suppose that $P_q(G)$ is not an l.c.e.-property. Then there exists a collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) of commutative semigroups S_γ such that $P_q(S_\alpha)$ is true for every multiple-component S_α of Ω and some composition $S(\circ)$ ($S = \Sigma\{S_\gamma : \gamma \in \Gamma\}$) of Ω is non-commutative. Hence, $a \circ b \neq b \circ a$ for some $a \in S_\gamma$, $b \in S_\beta$, $\gamma < \beta$. Now, define mappings ρ, $\sigma : S_\beta \to S_\beta$ as follows: $\rho(x) = a \circ x$ and $\sigma(x) = x \circ a$ for $x \in S_\beta$. Then ρ, σ are translations on S_β. For any $x \in S_\beta$, $\rho\sigma(x) = \sigma(\rho(x)) = \sigma(a \circ x) = (a \circ x) \circ a = a \circ (x \circ a) = a \circ \sigma(x) = \rho(\sigma(x)) = \sigma\rho(x)$. Thus $\rho\sigma = \sigma\rho$. Further for any $x, y \in S_\beta$, $\rho(xy) = a \circ xy = (a \circ x) \circ y = (a \circ x)\, y = y(a \circ x) = (y \circ a) \circ x = (y \circ a)\, x = x(y \circ a) = (x \circ y) \circ a = xy \circ a = \sigma(xy)$. Hence $\rho \mid S_\beta{}^2 = \sigma \mid S_\beta{}^2$. This contradicts to the assumption that S_β satisfies the property $P_q(G)$. Hence, $P_q(G)$ must be an l.c.e.-property. Next, suppose that there exists an l.c.e.-property $P(G)$ such that $P(G) \not\leqslant P_q(G)$. Then there exists a commutative semigroup S_0 such that $P(S_0)$ is true and $P_q(S_0)$ is not true. Since $P_q(S_0)$ is not true, there is a system $\{\sigma, \rho\}$ of distinct two translations σ, ρ on S_0 such that $\sigma\rho = \rho\sigma$ and $\sigma \mid S_0{}^2 = \rho \mid S_0{}^2$. Accordingly, there exist elements x, y, t of S_0 which satisfy (1), (2) of Lemma 9. Now, let $S_1 = (a)$ be an infinite cyclic semigroup generated by a and let $L = \{0, 1\}$ be a chain with respect to the usual multiplication. Define multiplication $\circ$ in $S = S_1 \dotplus S_0$ as follows: $a^i \circ u = \rho^i(u)$ for $u \in S_0$; $u \circ a^i = \sigma^i(u)$ for $u \in S_0$; $v \circ w = vw$ if $v, w \in S_1$ or $\in S_0$. Then $S(\circ)$ is a composition of $\{S_1, S_0\}$ with respect to L. Since $a \circ t = \rho(t) = y \neq x = \sigma(t) = t \circ a$, $S(\circ)$ is non-commutative. This contradicts to the assumption that $P(G)$ is an l.c.e.-property. Hence $P(G) \leqslant P_q(G)$.

COROLLARY. *Each of reductivity, universality and the property "reductive or universal" is an* l.c.e.*-property.*

Proof. In general, it is easy to see that if $P(G)$ and $P_1(G)$ are abstract properties such that $P_1(G) \leqslant P(G)$, and if $P(G)$ is an l.c.e.-property, then $P_1(G)$ is also an l.c.e.-property. For abstract properties $P_1(G)$ and $P_2(G)$, denote the property "$P_1(G)$ or $P_2(G)$" by $P_1(G) \vee P_2(G)$. It is obvious that $P_1(G) \leq P_1(G) \vee P_2(G)$ and $P_2(G) \leq P_1(G) \vee P_2(G)$. Now, we prove that $P_r(G) \vee P_u(G) \leq P_q(G)$. Suppose that $P_r(S) \vee P_u(S)$ is true for a commutative semigroup S. Since $P_r(S) \vee P_u(S)$ is true, at least one of $P_r(S)$ and $P_u(S)$ is true. If $P_q(S)$ is not true, then there is a system $\{\sigma, \rho\}$ of distinct two translations σ, ρ on S such that $\sigma\rho = \rho\sigma$ and $\sigma \mid S^2 = \rho \mid S^2$. Hence, by Lemma 9, there exist $x, y, t \in S$ which satisfy the conditions (1), (2) of Lemma 9. This contradicts to the assumption that S is reductive or universal. Hence, $P_q(S)$ must be true. Since $P_r(G) \leq P_r(G) \vee P_u(G) \leq P_q(G)$ and $P_u(G) \leq P_r(G) \vee P_u(G) \leq P_q(G)$ and since $P_q(G)$ is an l.c.e.-property, each of $P_r(G)$, $P_u(G)$ and $P_r(G) \vee P_u(G)$ is also an l.c.e.-property.

Remarks. (1) Moreover, the following is obvious from Theorem 4: Let $\Omega = \{S_\xi : \xi \in \chi\}$ (χ: a set) be a collection of commutative semigroups S_ξ, where $P_q(S_\xi)$ is true for all $S_\xi \in \Omega$. Then, every linear composition of Ω is commutative.

(2) For a special collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) of commutative semigroups S_γ, every composition of Ω is commutative even if there exists a multiple-component S_α which does not satisfy $P_q(G)$. For example, let $L = \{0, 1\}$ be a chain with respect to the usual multiplication, $S_1 = \{e\}$ a semigroup consisting of a single element e and $S_0 = \{a, a^2, \ldots, a^{n-1}, a^n\}$ a cyclic semigroup of order n ($n > 2$) such that $a^{n-1} \neq a^n$ and $aa^n = a^n$. Then, it is easy to see from the above-mentioned example that $P_q(S_0)$ is not true. However, there is no non-commutative composition of $\{S_1, S_0\}$ with respect to L.

Hereafter, for any element x of a commutative semigroup S, the inner translation on S induced by x will be denoted by ρ_x.

Now, let us consider the following abstract property $P_r^*(G)$ pertaining to commutative semigroups G:

(3.2) There is no system $\{u, v; \xi, \eta\}$ of distinct two elements u, v of G and (not necessarily distinct) translations ξ, η on G such that (1) $\xi\eta = \eta\xi = \rho_u = \rho_v$ and (2) $\xi(u) = \xi(v)$ and $\eta(u) = \eta(v)$.

LEMMA 10. *Let $\{u, v; \xi, \eta\}$ be a system of distinct two elements u, v of a commutative semigroup S and translations ξ, η on S, satisfying* (1), (2) *of* (3.2). *Then, $uz = vz$ for all $z \in S$.*

Proof. Since $\xi\eta = \eta\xi = \rho_u = \rho_v$, it follows that $\rho_u(z) = \rho_v(z)$, i.e. $uz = vz$ for all $z \in S$.

LEMMA 11. *$P_r^*(G)$ is equivalent to $P_r(G)$.*

Proof. At first, we prove that $P_r(G) \leq P_r^*(G)$. Suppose that $P_r^*(S)$ is not true for some commutative semigroup S. Then by Lemma 10, there exist distinct two elements u, v of S such that $uz = vz$ for all $z \in S$. Hence S is not reductive, and accordingly $P_r(S)$ is not true. Thus, $P_r(G) \leqslant P_r^*(G)$. Conversely, suppose that $P_r(S)$ is not true for some commutative semigroup S. Then, there exist $u, v \in S$ such that $uz = vz$ for all $z \in S$.

The case $S^2 \ni u$ or $\ni v$. Assume that $S^2 \ni u$, without the loss of generality. Let $u = cd$, $c, d \in S$, and let $\xi = \rho_c$ and $\eta = \rho_d$. Then $\xi\eta = \rho_c\rho_d = \rho_{cd} = \rho_u = \rho_v$ and $\eta\xi = \rho_d\rho_c = \rho_{dc} = \rho_{cd} = \rho_u = \rho_v$. Further, $\xi(u) = \rho_c(u) = uc = vc = \rho_c(v) = \xi(v)$ and $\eta(u) = \rho_d(u) = ud = vd = \rho_d(v) = \eta(v)$. Hence, there exists a system $\{u, v;\ \xi, \eta\}$ which satisfies (1), (2) of (3.2).

The case $S^2 \not\ni u, v$. Let $\xi = \rho_u$, and let η be a mapping: $S \to S$ such that $\eta(u) = v$, $\eta(v) = v$ and $\eta(x) = x$ if $x \neq u, v$. It is easy to see that η is a translation on S. (Note that $ux \neq u, v$ and $vx \neq v, u$ for all $x \in S$). At first, $\xi\eta = \rho_u$. In fact, $\rho_u\eta(u) = \eta(\rho_u(u)) = \eta(u^2) = uu = \rho_u(u)$, $\rho_u\eta(v) = \eta(\rho_u(v)) = \eta(uv) = uv = vu = \rho_u(v)$, and $\rho_u\eta(x) = \eta(\rho_u(x)) = \eta(xu) = xu = \rho_u(x)$ if $x \neq u, v$. Hence $\rho_u = \rho_u\eta = \xi\eta$. Next, $\eta\xi = \rho_u$. In fact, $\eta\rho_u(u) = \rho_u(\eta(u)) = \rho_u(v) = vu = uv = uu = \rho_u(u)$, $\eta\rho_u(v) = \rho_u(\eta(v)) = \rho_u(v)$, and $\eta\rho_u(x) = \rho_u(\eta(x)) = \rho_u(x)$ if $x \neq u, v$. Hence $\rho_u = \eta\rho_u = \eta\xi$. Consequently, we have $\xi\eta = \rho_u = \rho_v = \eta\xi$. Further, $\xi(u) = \rho_u(u) = uu = vu = \rho_u(v) = \xi(v)$, and $\eta(u) = v = \eta(v)$. Thus, the system $\{u, v;\ \xi, \eta\}$ satisfies (1), (2) of (3.2). This means that $P_r^*(S)$ is not true. Hence, it follows that $P_r^*(G) \leqslant P_r(G)$.

Since $P_r^*(G) \leqslant P_r(G)$ and $P_r(G) \leqslant P_r^*(G)$, we have $P_r^*(G) = P_r(G)$.

By using Lemmas 10 and 11, we can prove the following theorem which is one of the main results of this paper:

THEOREM 5. *$P_r(G)$ is the weakest* f.c.e.*-property.*

Proof. At first, we prove that $P_r(G)$ is an f.c.e.-property. Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) be a collection of commutative semigroups S_γ such that every multiple-component S_ξ satisfies $P_r(G)$. Let $S(\circ)$ be any composition of Ω. For any $x, y \in S(\circ)$, we next prove that $x \circ y = y \circ x$. Let $x \in S_\alpha$ and $y \in S_\beta$.

Case 1. ($\alpha = \beta$). In this case, $x \circ y = xy = yx = y \circ x$.

Case 2. ($\alpha \neq \beta$). In this case, $P_r(S_{\alpha\beta})$ is true since $S_{\alpha\beta}$ is a multiple-component. For $a, b \in S_{\alpha\beta}$, we have $(x \circ y)\, ab = x \circ ((y \circ a) \circ b) = x \circ ((y \circ a)\, b) = x \circ (b(y \circ a)) = x \circ (b \circ (y \circ a)) = (x \circ b) \circ (y \circ a) = (x \circ b)(y \circ a)$. Similarly, we have $(y \circ x)\, ab = a(y \circ x)\, b = (a \circ y)(x \circ b) = (x \circ b)(a \circ y) = ((x \circ b)\, a) \circ y = a((x \circ b) \circ y) = ((x \circ b) \circ y)\, a = (x \circ b) \circ (y \circ a) = (x \circ b)(y \circ a)$. Hence, $((x \circ y)\, a)\, b = ((y \circ x)\, a)\, b$ for all $a, b \in S_{\alpha\beta}$. Hence, by the reductivity of $S_{\alpha\beta}$, we have $x \circ y = y \circ x$.

Thus, $S(\circ)$ is commutative. Therefore, $P_r(G)$ is an f.c.e-property. Next, suppose that there exists an f.c.e.-property $P(G)$ such that $P(G) \nleq P_r(G)$. Since $P_r(G) \equiv P_r^*(G)$, $P(G) \nleq P_r^*(G)$. Hence, there is a commutative semigroup S_γ such that $P(S_\gamma)$ is true and $P_r^*(S_\gamma)$ is not true. Since $P_r^*(S_\gamma)$ is not true, there is a system $\{u, v;\ \xi, \eta\}$ of distinct elements u, v of S_γ and translations ξ, η on S_γ such that $\xi\eta = \eta\xi = \rho_u = \rho_v$, and $\xi(u) = \xi(v)$ and $\eta(u) = \eta(v)$. Let S_α and S_β be infinite cyclic semigroups generated by a and b respectively: $S_\alpha = (a)$ and $S_\beta = (b)$. Let $L_3 = \{\alpha, \beta, \gamma\}$ be a semilattice consisting of three elements α, β, γ such that $\alpha < \gamma, \beta < \gamma, \alpha \nleq \beta$ and $\beta \nleq \alpha$. Let $S = S_\alpha \dotplus S_\beta \dotplus S_\gamma$. Define multiplication $\circ$ in S as follows: $a^i \circ a^j = a^{i+j}; b^i \circ b^j = b^{i+j}; a \circ b = u, b \circ a = v; x \circ y = xy$ if $x, y \in S_\gamma$; $a^i \circ x = x \circ a^i = \xi^i(x)$ for $x \in S_\gamma$; $b^i \circ x = x \circ b^i = \eta^i(x)$ for $x \in S_\gamma$; and $a^i \circ b^j = \xi^{i-1}\eta^{j-1}(u)$, $b^j \circ a^i = \xi^{i-1}\eta^{j-1}(u)$ if $i + j \geqslant 3$ (we regard each of ξ^0 and η^0 as the identity mapping on S_γ). Then $S(\circ)$ is a non-commutative composition of $\{S_\alpha, S_\beta, S_\gamma\}$ with respect to L_3. This contradicts to the assumption that $P(G)$ is an f.c.e.-property and $P(S_\gamma)$ is true. Hence, $P_0(G) \leqslant P_r(G)$ for all f.c.e.-property $P_0(G)$.

COROLLARY. *Let $\Omega = \{S_\xi : \xi \in \chi\}$ (χ: a set) be a collection of commutative semigroups S_ξ, where $P_r(S_\xi)$ is true for each $S_\xi \in \Omega$. Then, every semilattice composition of Ω is commutative.*

Proof. Obvious.

Remarks. (1) It is easy to see that if $P_0(G)$ is an f.c.e. [l.c.e.]-property and if $P(G)$ is an abstract property such that $P(G) \leqslant P_0(G)$, then $P(G)$ is also an f.c.e.[l.c.e.]-property. Let $\mathfrak{P}(Q) = \{P(G) : P(G)$ is an abstract property such that $P(G) \leqslant P_q(G)\}$ and $\mathfrak{P}(R) = \{P(G) : P(G)$ is an abstract property such that $P(G) \leqslant P_r(G)\}$. Then, $\mathfrak{P}(Q)$ and $\mathfrak{P}(R)$ are the set of all l.c.e.-properties and the set of all f.c.e.-properties respectively.

(2) As was shown in Section 1, there exists a universal commutative semigroup S which has a zero element 0 and whose annihilator A contains a non-zero element. Since $P_q(S)$ is true and $P_r(S)$ is not true, $P_q(G) \not\equiv P_r(G)$. Hence, $P_q(G) > P_r(G)$. This also means that quasi-reductivity does not imply reductivity.

(3) Since $P_r(G)$ is weaker than each of separativity (see [*2*]) and cancellativity, the following results immediately follow from the above-mentioned Corollary:

(i) *A* semigroup which is a semilattice of commutative reductive semigroups is commutative and reductive.

(ii) A semigroup which is a semilattice of separative commutative semigroups is separative and commutative.

(iii) A semigroup which is a semilattice of cancellative commutative semigroups is separative and commutative.

The conserve of the result (iii) also holds (see [*2*]); i.e., a separative commutative semigroup is a semilattice of cancellative commutative semigroups.

4. Maximal F.C.I. [L.C.I]-Properties

In this section, we shall show the existence of a maximal l.c.i.-property and a maximal f.c.i.-property. Especially, it will be proved that $P_q(G)$ is a maximal l.c.i.-property and both $P_r(G)$ and $P_u(G)$ are maximal f.c.i.-properties.

Theorem 6. *There exist a maximal l.c.i.-property and a maximal f.c.i.-property.*

Proof. Let $\mathscr{F} = \{P_\lambda(G) : \lambda \in \Lambda\}$ be the set of all f.c.i.-properties $P_\lambda(G)$. Then $\mathscr{F}$ is clearly a partially ordered set with respect to the ordering relation $\leqslant$ defined by (1.6). (Recall that equivalent properties are regarded as the same property). Let $\mathscr{T} = \{P_\tau(G) : \tau \in \Lambda_0\}$ be any totally ordered subset of $\mathscr{F}$. Define an abstract property $T(G)$ as follows: $T(G) = \bigvee_{\tau\in\Lambda_0} P_\tau(G)$, i.e., $T(G) =$ the property "being at least one of $\{P_\tau(G) : \tau \in \Lambda_0\}$". Hence, a commutative semigroup S satisfies $T(G)$ if, and only if, S satisfies at least one of the properties $P_\tau(G)$, $\tau \in \Lambda_0$. Now, let $\Omega = \{S_\xi : \xi \in \chi\}$ (χ: a set) be a collection of commutative semigroups S_ξ such that every S_ξ satisfies $T(G)$. Suppose that there exists a non-commutative semilattice composition $S(\circ) = \Sigma\{S_\xi : \xi \in \chi(*)\}$ of Ω. Then there exist a, b such that $a \in S_\gamma$, $b \in S_\delta$, $\gamma, \delta \in \chi$ and $a \circ b \neq b \circ a$. Clearly, both $a \circ b$ and $b \circ a$ are contained in $S_{\gamma*\delta}$. Put $S_\gamma \dotplus S_\delta \dotplus S_{\gamma*\delta} = M$. Then $M(\circ)$ is a subsemigroup of $S(\circ)$ and is non-commutative. Since $T(S_\gamma)$, $T(S_\delta)$ and $T(S_{\gamma*\delta})$ are all true, there exist $P_\alpha(G)$, $P_\beta(G)$ and $P_\epsilon(G)$ in the collection $\{P_\tau(G) : \tau \in \Lambda_0\}$ such that $P_\alpha(S_\gamma)$, $P_\beta(S_\delta)$ and $P_\epsilon(S_{\gamma*\delta})$ are true. Let $P_\eta(G)$ be the weakest property in $\{P_\alpha(G)$, $P_\beta(G)$, $P_\epsilon(G)\}$. Then $P_\eta(G)$ is of course an f.c.i.-property and $P_\eta(S_\gamma)$, $P_\eta(S_\delta)$, $P_\eta(S_{\gamma*\delta})$ are all true. Hence, the semilattice composition $M(\circ)$ of $\{S_\gamma, S_\delta, S_{\gamma*\delta}\}$ must be commutative. However, this is a contradiction since $M(\circ)$ was non-communative. Consequently, every semilattice compostion of Ω must be commutative. Therefore, $T(G)$ is an f.c.i.-property and hence $T(G) \in \mathscr{F}$. Since $P_\tau(G) \leqslant T(G)$ for all $\tau \in \Lambda_0$, $T(G)$ is an upper bound of $\mathscr{T}$. Thus, $\mathscr{F}$ is an inductively ordered set. Hence, there exists a maximal f.c.i.-property in $\mathscr{F}$. The existence of a maximal l.c.i.-property is also proved by a similar method.

COROLLARY. *For any* f.c.i. [l.c.i.]-*property* $P(G)$, *there exists a maximal* f.c.i. [l.c.i.]-*property* $P_m(G)$ *such that* $P(G) \leqslant P_m(G)$.

Proof. This can be proved by an analogous method to Theorem 6.

In fact, the following three theorems show that quasi-reductivity is a maximal l.c.i.-property and both reductivity and universality are maximal f.c.i.-properties:

THEOREM 7. $P_q(G)$ *is a maximal* l.c.i.-*property.*

Proof. It is obvious from Remark (1) for Theorem 4 that $P_q(G)$ is an l.c.i.-property. Suppose that there is an l.c.i.-property $P(G)$ such that $P(G) > P_q(G)$. Then, there exists a commutative semigroup S_0 such that $P(S_0)$ is true and $P_q(S_0)$ is not true. Since $P_q(S_0)$ is not true, there is a system $\{\sigma, \rho\}$ of distinct two translations on S_0 such that $\sigma\rho = \rho\sigma$ and $\sigma \mid S_0{}^2 = \rho \mid S_0{}^2$. Hence by Lemma 9, there exist distinct two elements x, y and a prime element t in S_0 such that $xa = ya$ for all $a \in S_0$, $\sigma(t) = x$ and $\rho(t) = y$. Let $S_1 = (a)$ be an infinite cyclic semigroup generated by a, and let $S_1 \dotplus S_0 = S$. Define multiplication $\circ$ in S as follows: $a^i \circ u = \rho^i(u)$ for $u \in S_0$; $u \circ a^i = \sigma^i(u)$ for $u \in S_0$; $v \circ w = vw$ if $v, w \in S_1$ or $\in S_0$. Then $S(\circ)$ becomes a non-commutative linear composition of $\{S_1, S_0\}$. Since $P_q(S_1)$ is true and $P_q(G) < P(G)$, $P(S_1)$ is also true. Hence both S_1 and S_0 satisfy $P(G)$, but there is a non-commutative linear composition of $\{S_1, S_0\}$. Thus, we have a contradiction. Hence, there is no l.c.i.-property $P(G)$ such that $P(G) > P_q(G)$.

THEOREM 8. $P_r(G)$ *is a maximal* f.c.i.-*property.*

Proof. It is obvious from Corollary to Theorem 5 that $P_r(G)$ is an f.c.i.-property. Suppose that there is an f.c.i.-property $P(G)$ such that $P_r(G) < P(G)$. Then, there exists a commutative semigroup S_γ such that $P(S_\gamma)$ is true and $P_r(S_\gamma)$ is not true. Since $P_r(S_\gamma)$ is not true, there is a system $\{u, v; \xi, \eta\}$ of distinct two elements u, v of S_γ and translations ξ, η on S_γ such that $\xi\eta = \eta\xi = \rho_u = \rho_v$, $\xi(u) = \xi(v)$ and $\eta(u) = \eta(v)$. Let S_α and S_β be infinite cyclic semigroups generated by a and b respectively: $S_\alpha = (a)$ and $S_\beta = (b)$. Define multiplication $\circ$ in $S = S_\alpha \dotplus S_\beta \dotplus S_\gamma$ as follows: $a^i \circ a^j = a^{i+j}$; $b^i \circ b^j = b^{i+j}$; $a \circ b = u, b \circ a = v$; $x \circ y = xy$ if $x, y \in S_\gamma$; $a^i \circ x = x \circ a^i = \xi^i(x)$ for $x \in S_\gamma$; $b^i \circ x = x \circ b^i = \eta^i(x)$ for $x \in S_\gamma$; and $a^i \circ b^j = b^j \circ a^i = \xi^{i-1}\eta^{j-1}(u)$ if $i + j \geqslant 3$ (where ξ^0, η^0 are regarded as the identity mapping on S_γ). Then $S(\circ)$ becomes a non-commutative semilattice composition of $\{S_\alpha, S_\beta, S_\gamma\}$. Since $P_r(S_\alpha)$ and $P_r(S_\beta)$ are true and $P_r(G) < P(G)$, $P(S_\alpha)$ and $P(S_\beta)$ are also true. Hence, the semilattice composition $S(\circ)$ of $\{S_\alpha, S_\beta, S_\gamma\}$ must be commutative since $P(G)$ is an f.c.i.-property and since each of S_α, S_β and S_γ satisfies $P(G)$. Thus, we have a contradiction. Therefore, there is no f.c.i.-property $P(G)$ such that $P_r(G) < P(G)$.

THEOREM 9. *$P_u(G)$ is a maximal* f.c.i.*-property.*

Proof. At first, we prove that universality $P_u(G)$ is an f.c.i.-property. Let $\Omega = \{S_\xi : \xi \in \chi\}$ (χ: a set) be any collection of universal commutative semigroups S_ξ. Let $S(\circ) = \Sigma\{S_\xi : \xi \in \chi(*)\}$ be a semilattice composition of Ω. Let a, b be any two elements of $S = \Sigma\{S_\xi : \xi \in \chi\}$. Then, there exist S_α, $S_\beta \in \Omega$ such that $S_\alpha \ni a$ and $S_\beta \ni b$. It is obvious that $a \circ b \in S_{\alpha*\beta}$. Since $a \in S_\alpha$ and S_α is universal, $a = a_1a_2$ for some a_1, $a_2 \in S_\alpha$. Now, clearly each of $a_1 \circ b$, $b \circ a_1$, $a_2 \circ b$ and $b \circ a_2$ is contained in $S_{\alpha*\beta}$. $a \circ b = a_1a_2 \circ b = a_1 \circ (a_2 \circ b) = a_1 \circ xy$ for some x, $y \in S_{\alpha*\beta}$. $a_1 \circ xy = (a_1 \circ x)y = y \circ (a_1 \circ x) = (y \circ a_1)x = x(y \circ a_1) = xy \circ a_1 = (a_2 \circ b) \circ a_1 = a_2 \circ (b \circ a_1) = a_2 \circ (b \circ a_1) = a_2 \circ zw$ for some $z, w \in S_{\alpha*\beta}$. Further, $a_2 \circ zw = (a_2 \circ z)w = w(a_2 \circ z) = (w \circ a_2)z = z(w \circ a_2) = zw \circ a_2 = (b \circ a_1) \circ a_2 = b \circ a_1a_2 = b \circ a$. Hence, we have $a \circ b = b \circ a$. This means that $S(\circ)$ is commutative. Thus, $P_u(G)$ is an f.c.i.-property. Next, suppose that there is an f.c.i.-property $P(G)$ such that $P_u(G) < P(G)$. Then, there exists a commutative semigroup S_0 such that $P(S_0)$ is true and $P_u(S_0)$ is not true. Let S_γ be a universal commutative semigroup which has a zero element 0 and whose annihilator A contains a nonzero element v. (The existence of such a semigroup S_γ has been shown in Remark of Section 1). Since $P_u(S_\gamma)$ is true and $P_u(G) < P(G)$, $P(S_\gamma)$ is also true. Put $S_\alpha = S_\beta = S_0$, and define multiplication $\circ$ in $S = S_\alpha \dotplus S_\beta \dotplus S_\gamma$ as follows: $x \circ y = xy$ if $x, y \in S_\alpha$, $\in S_\beta$ or $\in S_\gamma$; $S_\alpha \circ S_\gamma = S_\gamma \circ S_\alpha = S_\beta \circ S_\gamma = S_\gamma \circ S_\beta = \{0\}$; $S_\alpha{}^2 \circ S_\beta = S_\beta \circ S_\alpha{}^2 = S_\beta{}^2 \circ S_\alpha = S_\alpha \circ S_\beta{}^2 = \{0\}$; $S_\alpha^* \circ S_\beta^* = \{v\}$; and $S_\beta^* \circ S_\alpha^* = \{0\}$, where $S_\alpha^* = S_\alpha \backslash S_\alpha{}^2$, $S_\beta^* = S_\beta \backslash S_\beta{}^2$ and 0 is the zero element of S_γ. Then, the resulting system $S(\circ)$ is a non-commutative semilattice composition of Ω. This contradicts to the fact that $P(G)$ is an f.c.i.-property and each of S_α, S_β and S_γ satisfies $P(G)$. Hence, there is no f.c.i.-property $P(G)$ such that $P_u(G) < P(G)$. That is, $P_u(G)$ is a maximal f.c.i.-property.

From Theorem 9, we also have immediately

COROLLARY. *A semigroup which is a semilattice of universal commutative semigroups is universal and commutative.*

Remark. Let $\mathscr{F}[\mathscr{L}]$ be the set of all f.c.i. [l.c.i.]-properties. For $P_1(G)$, $P_2(G) \in \mathscr{F}[\mathscr{L}]$, let us define an abstract property $P_1(G) \wedge P_2(G)$ as follows:

(4.1) $P_1(G) \wedge P_2(G) =$ the property "being both $P_1(G)$ and $P_2(G)$". Then, it is easy to see that $P_1(G) \wedge P_2(G) \in \mathscr{F}[\mathscr{L}]$ for any $P_1(G)$, $P_2(G) \in \mathscr{F}[\mathscr{L}]$ and

$P_1(G) \wedge P_2(G)$ is the greatest lower bound of $P_1(G)$ and $P_2(G)$. Further, in fact $\mathscr{F}[\mathscr{L}]$ is a semilattice with respect to this operation $\wedge$.

Since $P_u(G)$ and $P_r(G)$ are non-equivalent maximal f.c.i.-properties, it is obvious that there is no greatest f.c.i.-property, i.e., there is no weakest f.c.i.-property $P_g(G)$ in the following sense:

(4.2) $P(G) \leqslant P_g(G)$ for any f.c.i.-property $P(G)$.

However, the authors can not solve the following two problems and leave them as open problems:

Problem 1. Is there a maximal l.c.i.-property except $P_q(G)$? That is: Is $P_q(G)$ the greatest (weakest) l.c.i.-property? Determine all of the maximal l.c.i.-properties.

Problem 2. Is there a maximal f.c.i.-property except $P_u(G)$ and $P_r(G)$? Determine all of the maximal f.c.i.-properties.

As a partial solution of Problem 1, we obtain the following result: Let C be an infinite cyclic semigroup: $C = \{a, a^2,..., a^n,...\}$. Let C^1 be the adjunction of an identity element to C: $C^1 = C \dotplus \{1\}$. Let $\mathscr{L}_0$ be the set of all l.c.i.-properties $L(G)$ satisfied by C^1, and $\mathscr{L}^*$ the set of all abstract properties $P(G)$ such that $P(G) \leqslant L(G)$ for some $L(G) \in \mathscr{L}_0$. Then $\mathscr{L}^* \ni P_q(G), P_r(G), P_u(G)$, since each of $P_q(C^1)$, $P_u(C^1)$ and $P_r(C^1)$ is true and each of $P_q(G)$, $P_u(G)$ and $P_r(G)$ is an l.c.i.-property. Further, $\mathscr{L}^* \supset \bar{\mathscr{L}} = \{P(G) : P(G)$ is an l.c.i.-property which is comparable with $P_r(G)$ or $P_u(G)\}$. In fact, let $P(G)$ be a property contained in $\bar{\mathscr{L}}$. If $P(G) \leqslant P_r(G)$ or $\leqslant P_u(G)$, then $P(G) \in \mathscr{L}^*$ since each of $P_r(G)$ and $P_u(G)$ is an l.c.i.-property and is satisfied by C^1. If $P(G) > P_r(G)$ or $> P_u(G)$, then $P(C^1)$ is true since each of $P_r(C^1)$ and $P_u(C^1)$ is true. Since $P(C^1)$ is true and $P(G)$ is an l.c.i.-property, $P(G)$ is also contained in $\mathscr{L}^*$. In any case, $P(G) \in \mathscr{L}^*$. Therefore, $\bar{\mathscr{L}} \subset \mathscr{L}^*$. Especially, cancellativity, separativity, regularity and the property "being a commutative semigroup G with 1" are all contained in $\mathscr{L}^*$.

Now, we have

THEOREM 10. *$P_q(G)$ is the greatest* (i.e. *weakest*) l.c.i.*-property in $\mathscr{L}^*$.*

Proof. Let $P(G)$ be an l.c.i.-property contained in $\mathscr{L}^*$. Then, there exists an l.c.i.-property $L(G)$ such that $L(C^1)$ is true and $P(G) \leqslant L(G)$. Suppose that $L(G) \nleqslant P_q(G)$. Then, there exists an commutative semigroup S_0 such that $L(S_0)$ is true and $P_q(S_0)$ is not true. Since $P_q(S_0)$ is not true, there exists a system $\{\sigma, \rho\}$ of distinct two translations σ, ρ on S_0 such that $\sigma\rho = \rho\sigma$ and $\sigma \mid S_0^2 = \rho \mid S_0^2$. Let $S_1 = C$ (= the infinite cyclic semigroup (a)), and define multiplication $\circ$ in $S^* = S_1 \dotplus S_0$ as follows: $a^i \circ u = \rho^i(u)$ for $u \in S_0$; $u \circ a^i = \sigma^i(u)$ for $u \in S_0$; and $v \circ w = vw$ if $v, w \in S_1$ or $\in S_0$.

Then, $S^*(\circ)$ is a non-commutative linear composition of $\{S_1, S_0\}$. Let S be the adjunction of an identity element to $S^*(\circ)$: $S = S^* \dotplus \{1\} = C^1 \dotplus S_0$. Then S is clearly a non-commutative linear composition of $\{C^1, S_0\}$. This contradicts to our assumption that $L(C^1)$, $L(S_0)$ are true and $L(G)$ is an l.c.i.-property. Hence $L(G) \leqslant P_q(G)$, and accordingly $P(G) \leqslant L(G) \leqslant P_q(G)$. This means that $P_q(G)$ is the greatest l.c.i.-property in $\mathscr{L}^*$.

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